Department of Computer Science, Yazd University

Delaunay Triangulations-Part II

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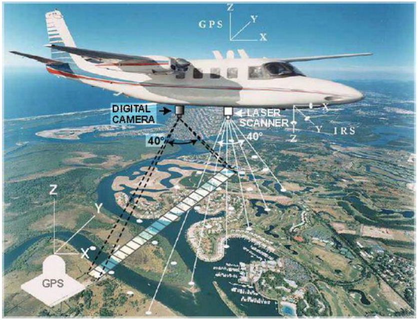
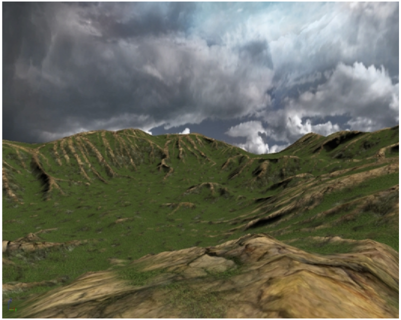
Review

Purpose: Approximating a terrain by constructing a polyhedral terrain from a set P of sample points.





Review

Theorem 9.1:

Let P $=\{p_1, p_2, ..., p_n\}$ be a point set. A triangulation of P is a maximal planar subdivision with vertex set P.

- ☞ triangles= $2n - 2 - k$ Back
- ☞ edges= $3n - 3 - k$

where k is the number of points in P on the convex hull of P

Theorem 9.2:(Thales Theorem)

Let C be a circle,L a line intersecting C in points a and b, and p,q,r and s points lying on the same side of L.Suppose that p and q lie on C, that r lies inside C,and that s lies outside C.Then

$$\angle arb > \angle abq = \angle aqb > \angle asb$$

Review

Observation 9.3:

Let T be a triangulation with an illegal edge e.Let T' be the triangulation obtained from T by flipping e.Then

$$A(T') > A(T)$$

Definition:

A legal triangulation is a triangulation that does not contain any illegal edge.

Conclusion:

Any angle-optimal triangulation is legal.

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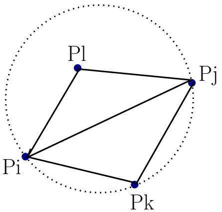
Let edge $\overline{p_ip_j}$ be incident to triangles $p_ip_jp_k$ and $p_ip_jp_l$ and let C be the circle through p_i,p_j and p_k. The edge $\overline{p_ip_j}$ is illegal if and only if the point p_l lies in the interior of C.

☞ if the points p_i,p_j,p_k,p_l from a convex quadrilateral and do not lie on a common circle $\Rightarrow$ exactly one of $\overline{p_ip_j}$ and $\overline{p_kp_l}$ is an illegal edge. Back



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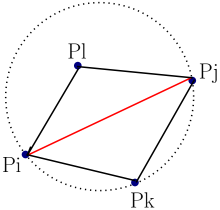

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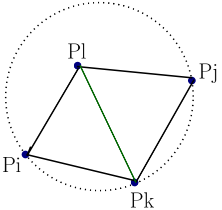
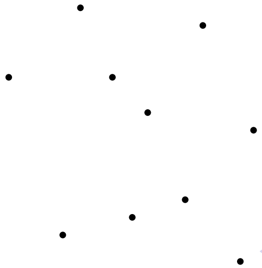


Review

Algorithm LEGALTRIANGULATION($\mathcal{T}$)
Input. A triangulation $\mathcal{T}$ of a point set P.
Output. A legal triangulation of P.
1. **while** $\mathcal{T}$ contains an illegal edge $\overline{p_ip_j}$
2. **do** (∗ Flip $\overline{p_ip_j}$ ∗)
3. Let $p_ip_jp_k$ and $p_ip_jp_l$ be the two triangles adjacent to $\overline{p_ip_j}$.
4. Remove $\overline{p_ip_j}$ from $\mathcal{T}$, and add $\overline{p_kp_l}$ instead.
5. **return** $\mathcal{T}$

Review

1. A set P of n points in the plane
2. The Voronoi diagram $Vor(P)$ is the subdivision of the plane into Voronoi cells $V(p)$ for all $p \in P$
3. Let G be the dual graph of $Vor(P)$
4. The Delaunay graph $DG(P)$ is the straight line embedding of G.



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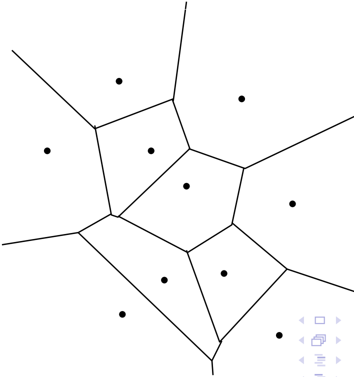

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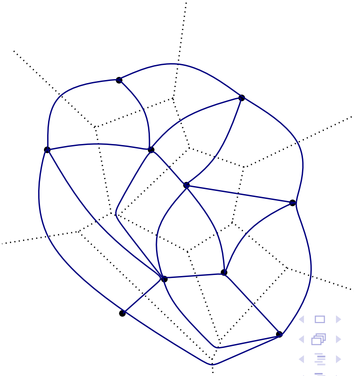

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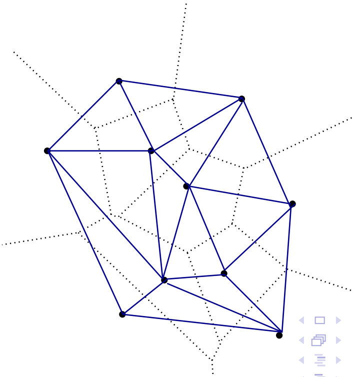


Back

Review

Lemma 9.5:

The Delaunay graph of a planar point set is a plane graph.

$\surd$

The edge $\overline{p_i p_j}$ is in the Delaunay graph $Dg(P) \Longleftrightarrow$ there is a C_ij whit p_i and p_j on its boundary and no other site of P contained in it.
The center of such a disc lies on the common edge of $V(p_i)$ and $V(p_j)$.

☞ If the point set P is in general position then the Delaunay graph is a triangulation.

Review

Theorem 9.6:

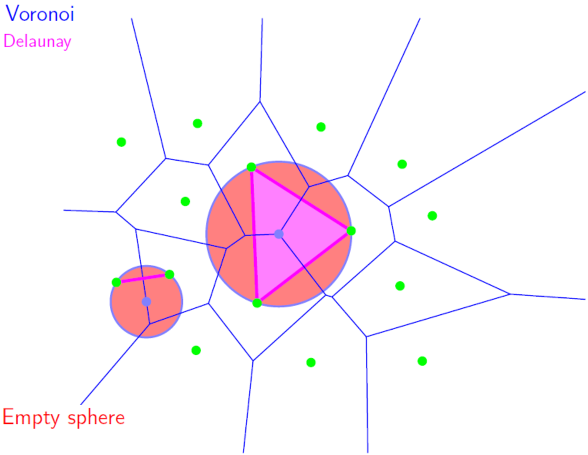
Back Let P be a set of points in the plane,

- Three points $p_i, p_j, p_k \in P$ are vertices of the same face of the Delaunay graph of $P \Longleftrightarrow$ the circle through p_i, p_j, p_k contains no point of P in its interior.
- Two points $p_i, p_j \in P$ form an edge of the Delaunay graph of $P \Longleftrightarrow$ there is a closed disc C that contains p_i and p_j on its boundary and does not contain any other point of P.

Theorem 9.7:

T is a Delaunay triangulation of $P \Longleftrightarrow$ the circumcircle of any triangle of T does not contain a point of P in its interior.

Review



Review

Theorem 9.8:

Let P be a set of points in the plane.A triangulation T of P is legal $\Longleftrightarrow$ T is a Delaunay triangulation P.

Theorem 9.9:

Let P be a set of points in the plane.Any angle-optimal triangulation of P is a Delaunay triangulation P. Furthermore,any Delaunay triangulation of P maximizes the minimum angle over all triangulations of$P$.

Review

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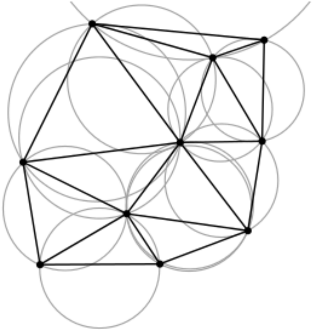
- A Delaunay triangulation for a set P of points in a plane is a triangulation $DT(P)$ such that no point in P is inside the circumcircle of any triangle in $DT(P)$.



Review

- Delaunay triangulations maximize the minimum angle of all the angles of the triangles in the triangulation; they tend to avoid skinny triangles.
- For a set of points on the same line there is no Delaunay triangulation (the notion of triangulation is degenerate for this case)
- For four or more points on the same circle (e.g., the vertices of a rectangle) the Delaunay triangulation is not unique
- By considering circumscribed spheres, the notion of Delaunay triangulation extends to three and higher dimensions

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Review

- The triangulation is named after Boris Delaunay for his work on this topic from 1934.

Review

- The voronoi diagram is named after Georgy F. Voronoi for his work on this topic.

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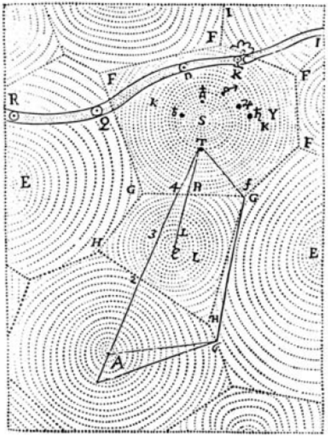


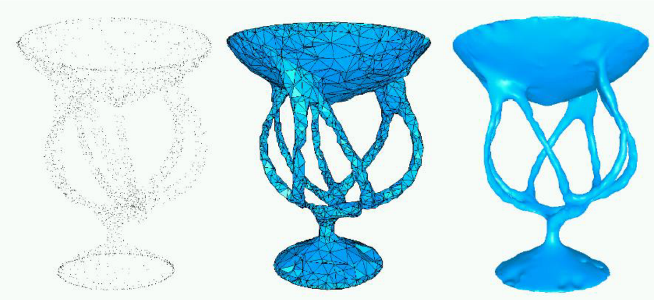
Usage

Delaunay triangulations help in constructing various things:

- Euclidean Minimum Spanning Trees
- Approximations to the Euclidean Traveling Salesperson Problem
- , ...

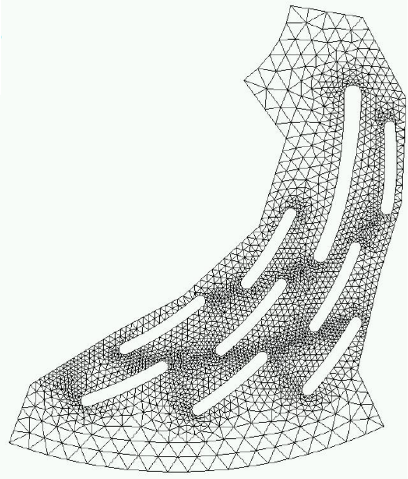
Usage

Reconstruction



Usage

Meshing



Usage

Meshing / Remeshing

Methods

There are several ways to compute the Delaunay triangulation:

- By plane sweep
- By iterative flipping from any triangulation
- By conversion from the Voronoi diagram Go to diagram
- By randomized incremental approach $\surd$

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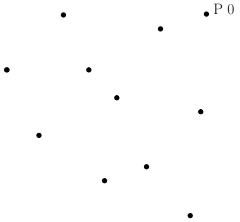

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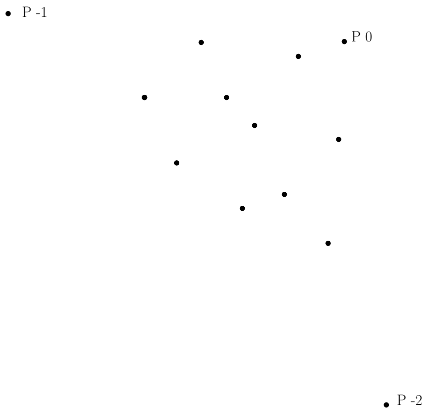

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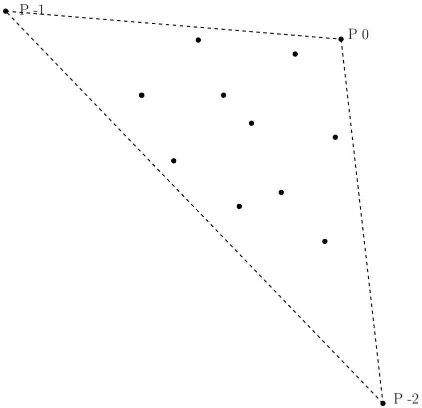

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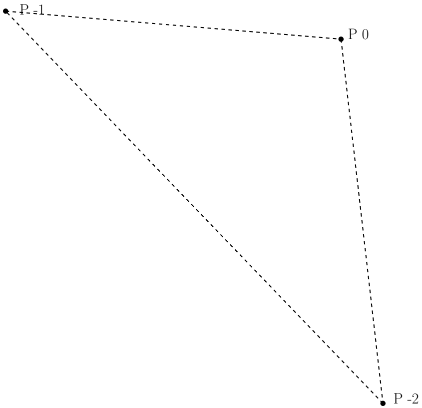

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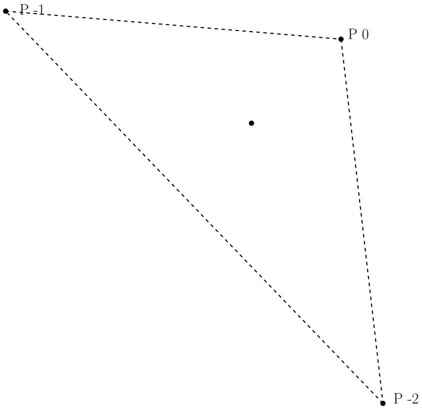

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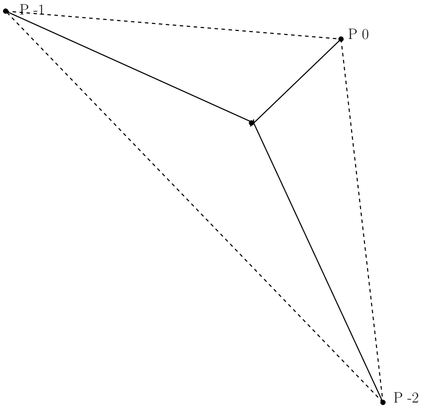

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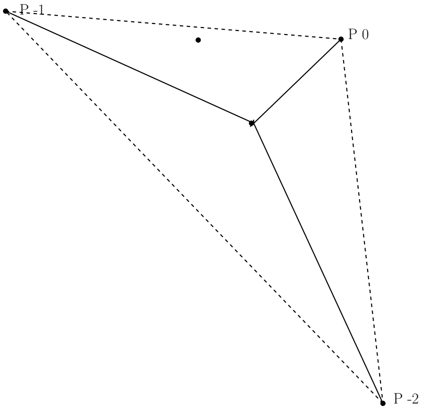

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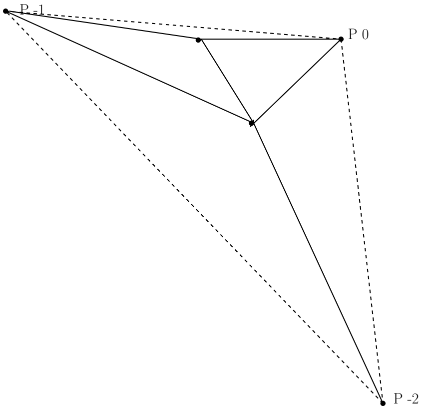

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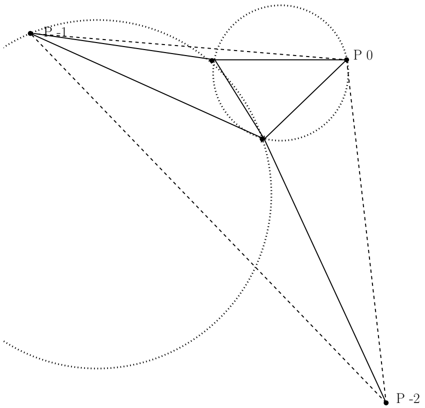

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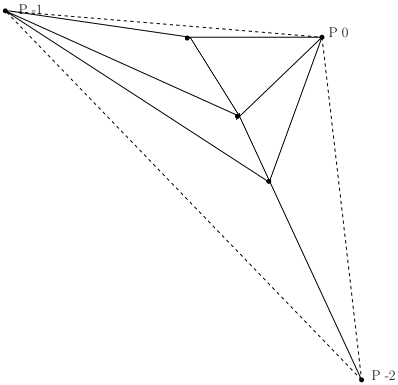

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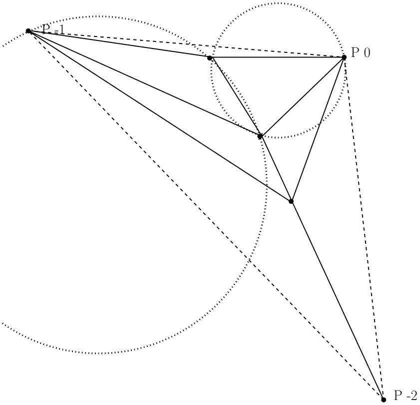

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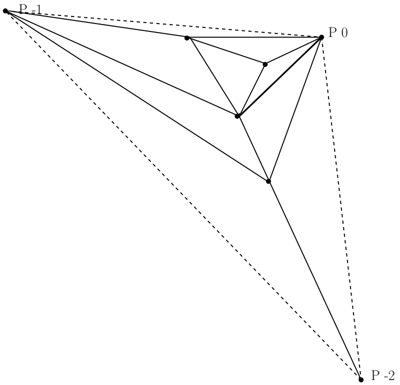

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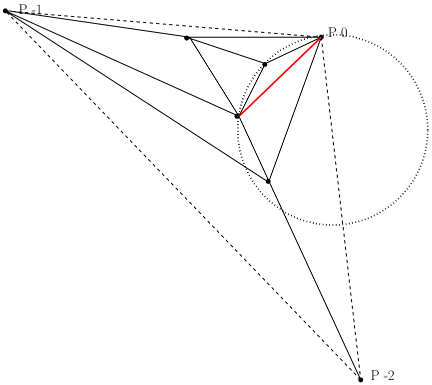

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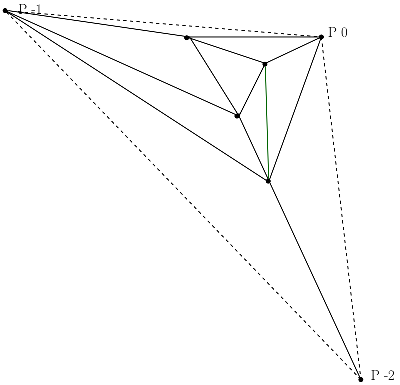

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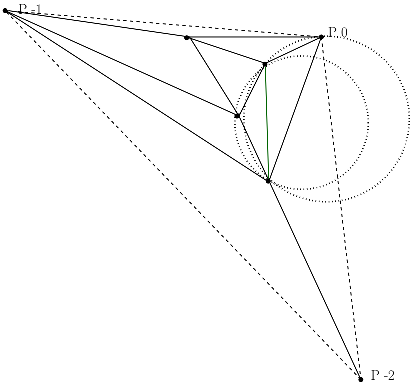

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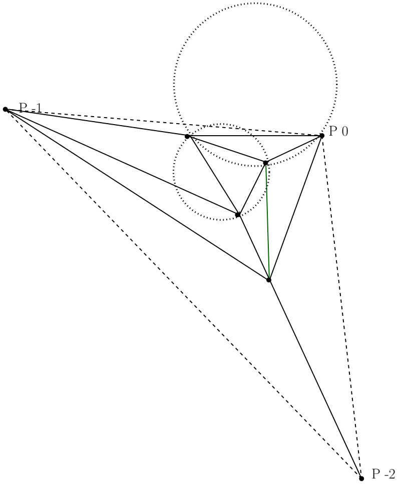

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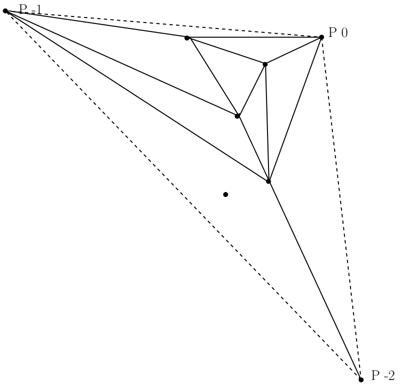

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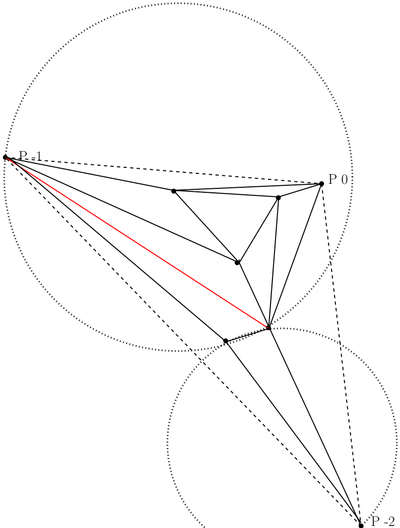

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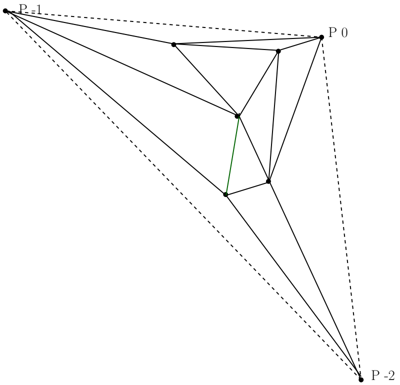

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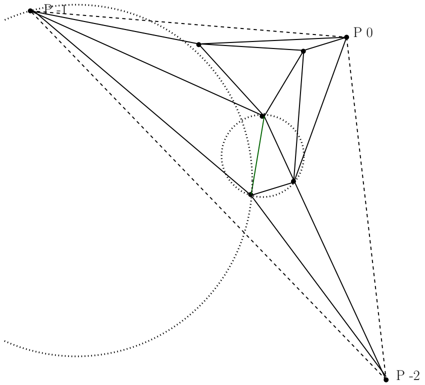

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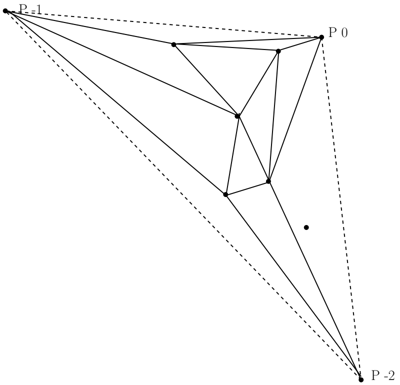

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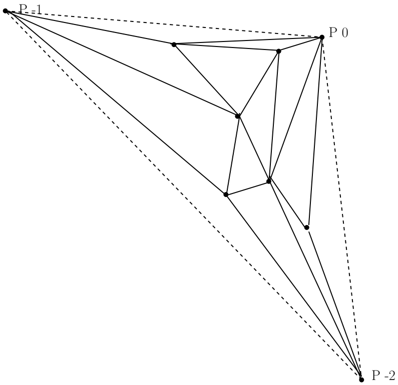

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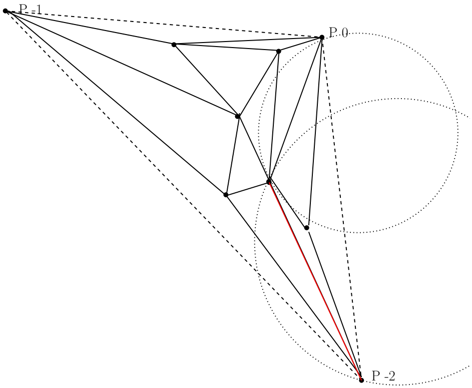

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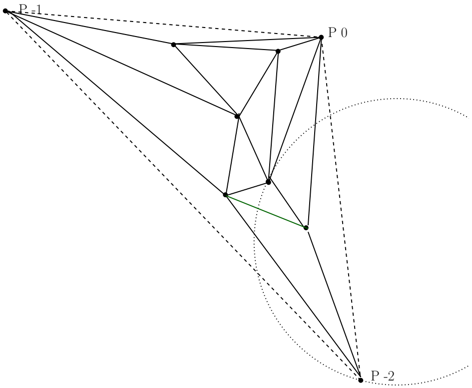

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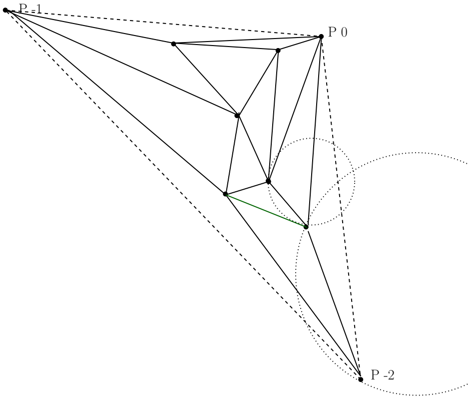

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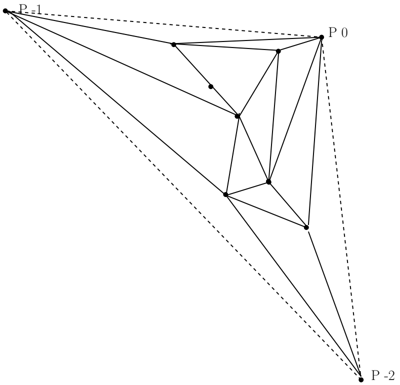

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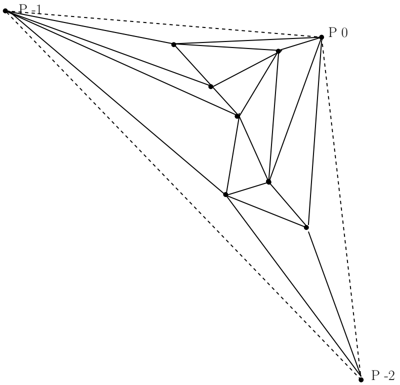

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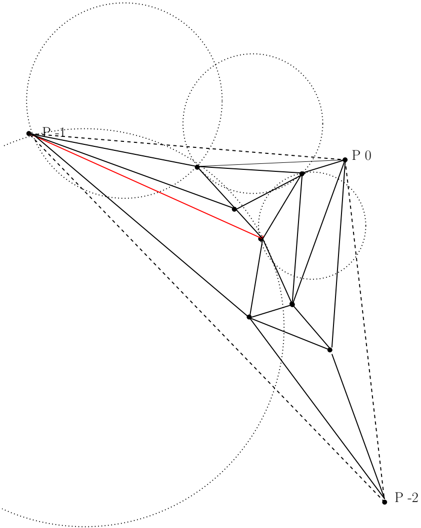

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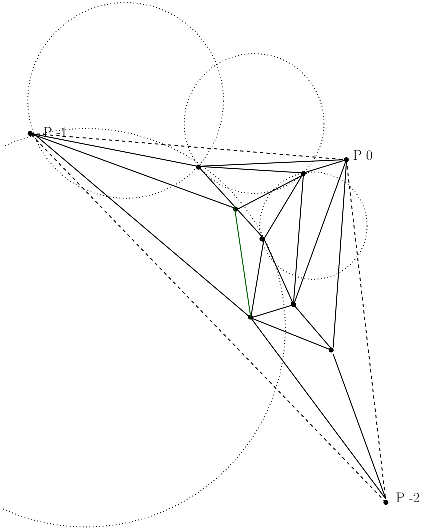

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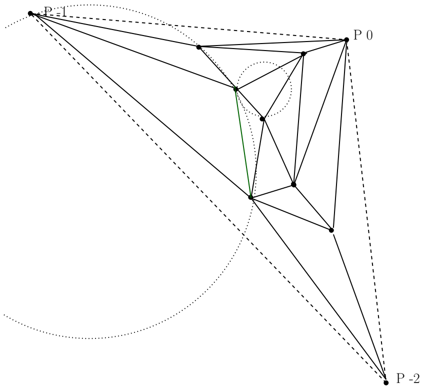

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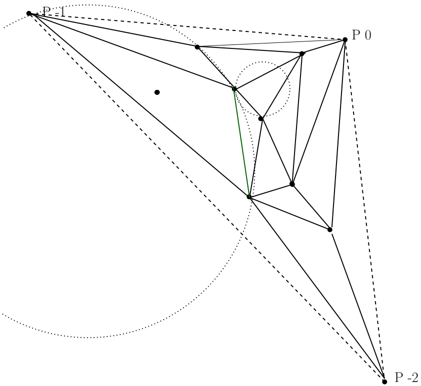

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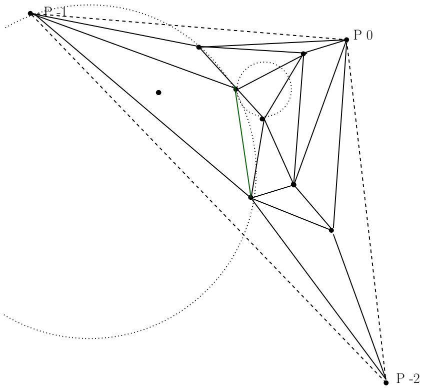

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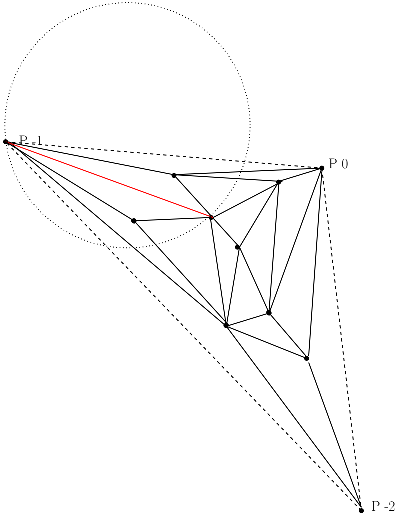

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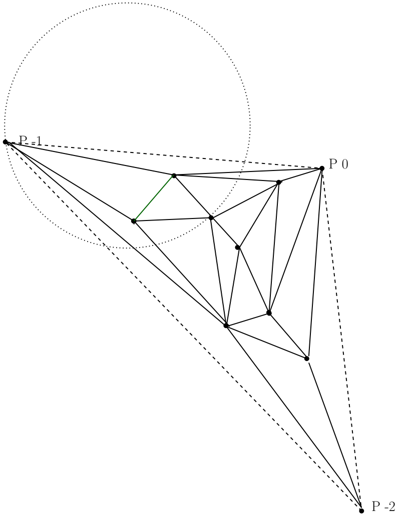

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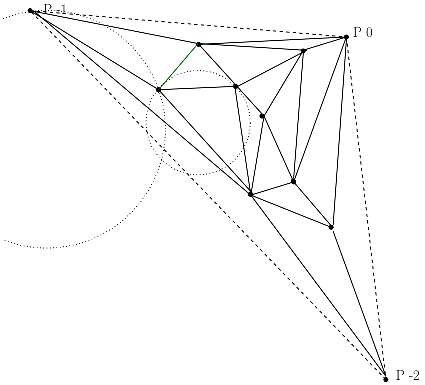

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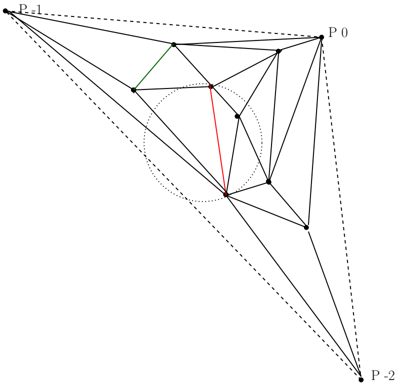

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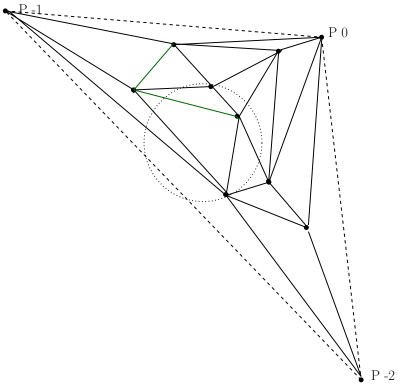

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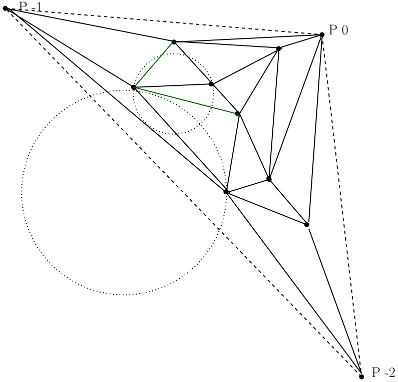

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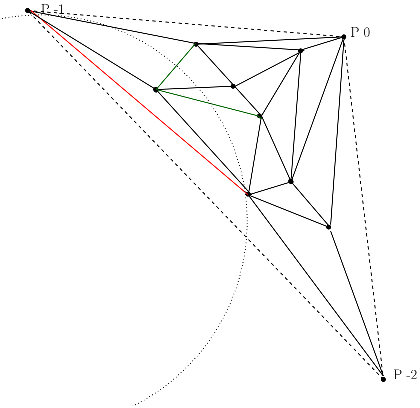

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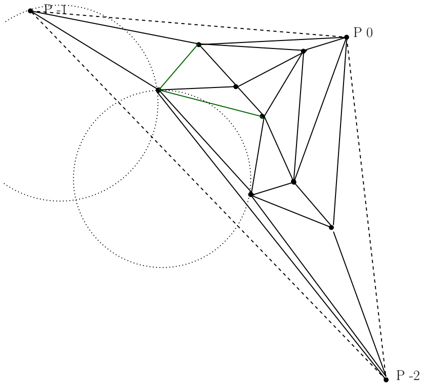

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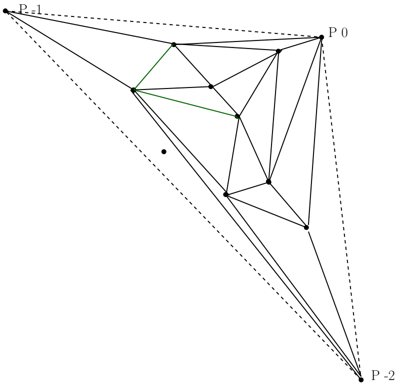

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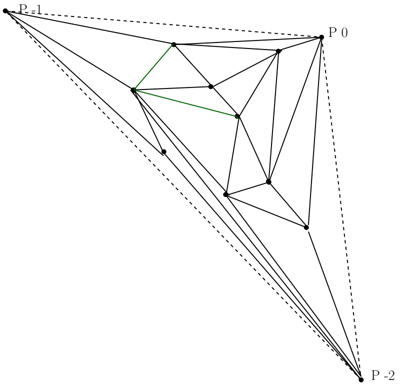

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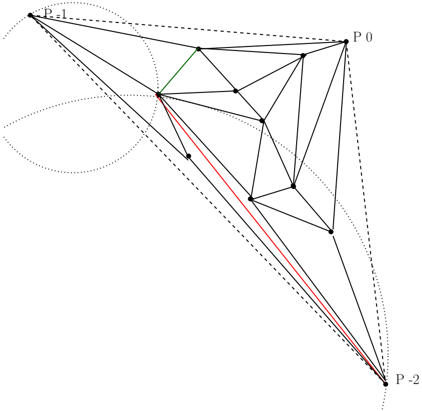

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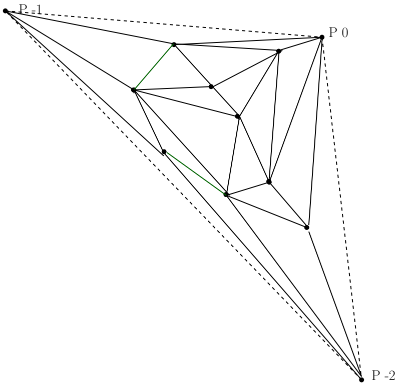

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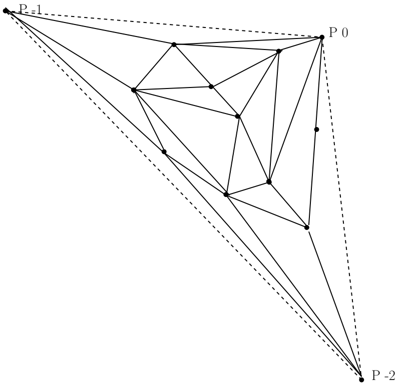

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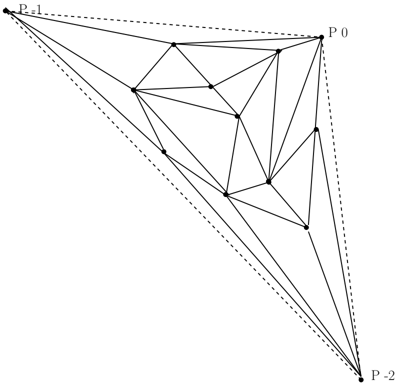

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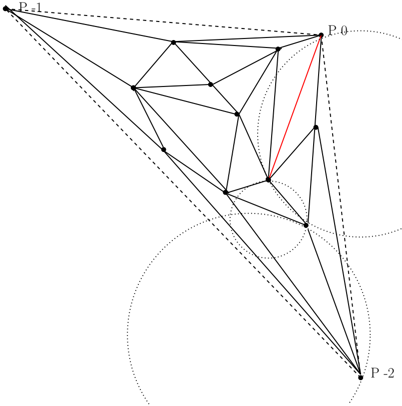

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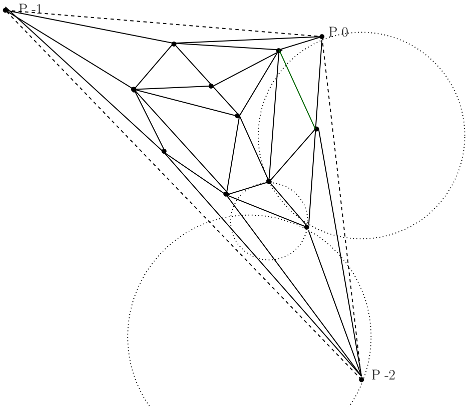

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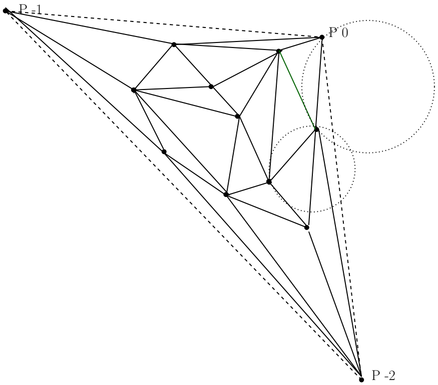

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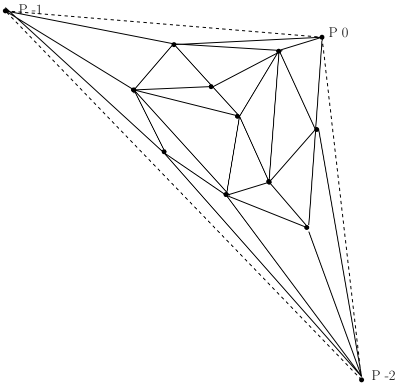

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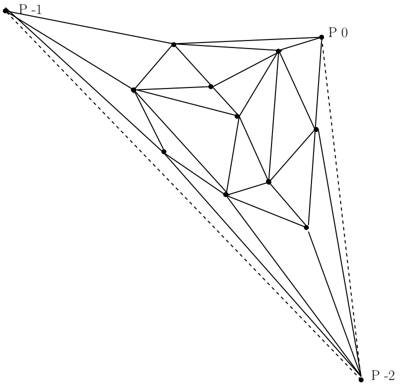

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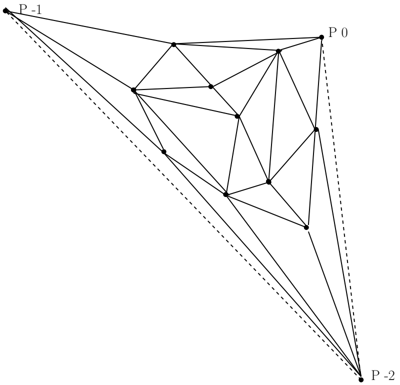

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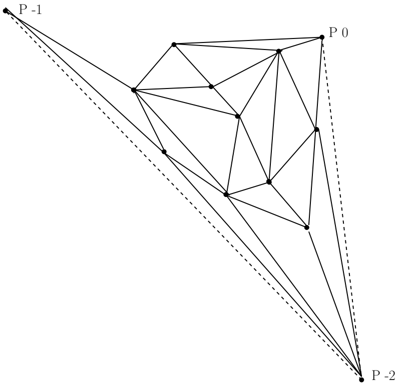

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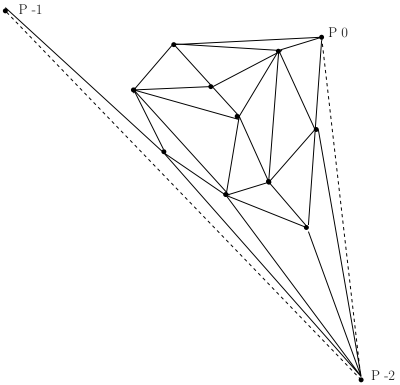

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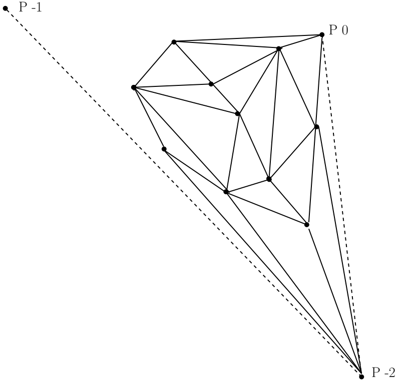

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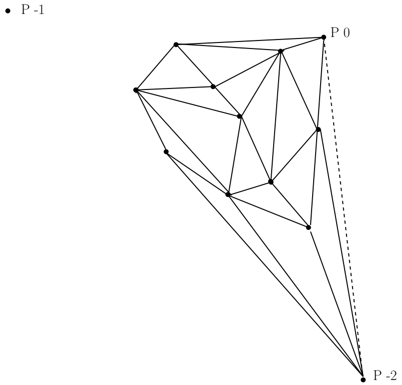

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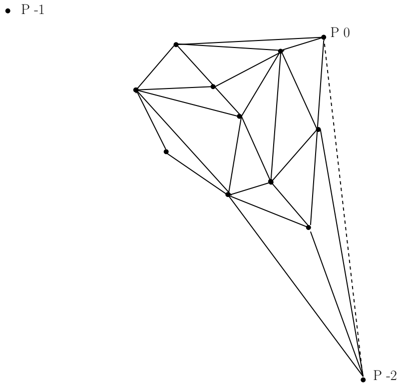

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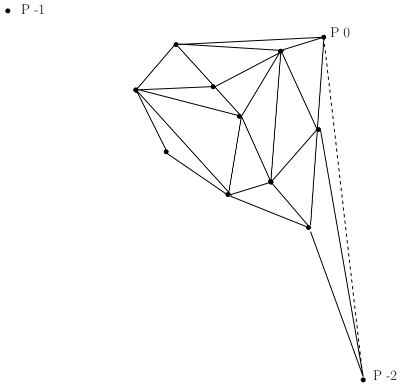

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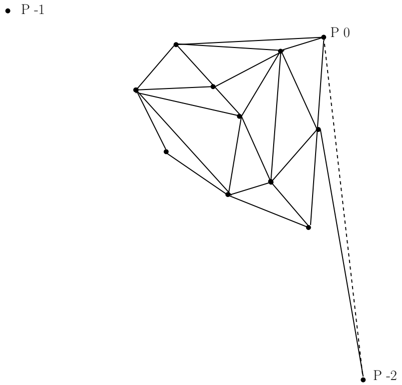

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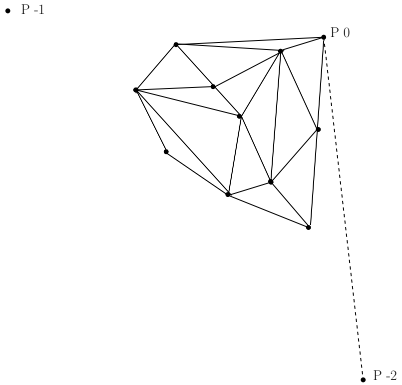

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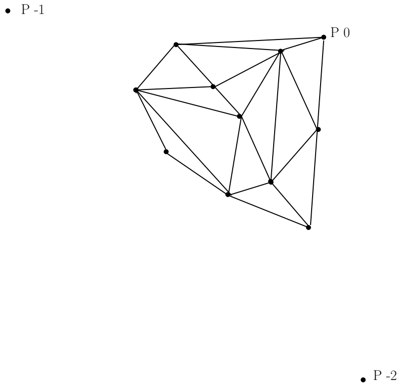

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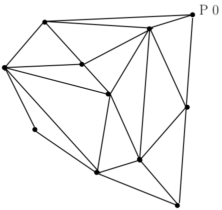

Pseudocode

Back

Algorithm DELAUNAYTRIANGULATION(P)
Input. A set P of $n+1$ points in the plane.
Output. A Delaunay triangulation of P.

Let p_0 be the lexicographically highest point of P, that is, the rightmost among the points with largest y-coordinate.
Let p_{-1} and p_{-2} be two points in $\mathbb{R}^2$ sufficiently far away and such that P is contained in the triangle $p_0p_{-1}p_{-2}$.
Initialize $\mathcal{T}$ as the triangulation consisting of the single triangle $p_0p_{-1}p_{-2}$.
Compute a random permutation $p_1, p_2, \ldots, p_n$ of $P \setminus \{p_0\}$.
for $r \leftarrow 1$ **to** n
    do (* Insert p_r into $\mathcal{T}$: *)
        Find a triangle $p_ip_jp_k \in \mathcal{T}$ containing p_r.
        if p_r lies in the interior of the triangle $p_ip_jp_k$
          then Add edges from p_r to the three vertices of $p_ip_jp_k$, thereby splitting $p_ip_jp_k$ into three triangles.
              LEGALIZEEDGE($p_r, \overline{p_ip_j}, \mathcal{T}$)
              LEGALIZEEDGE($p_r, \overline{p_jp_k}, \mathcal{T}$)
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for $r \leftarrow 1$ **to** n
    do (∗ Insert p_r into $\mathcal{T}$: ∗)
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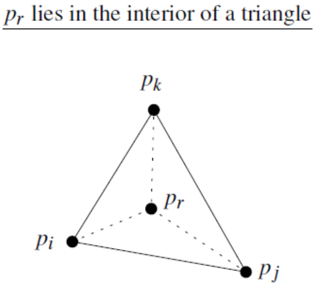
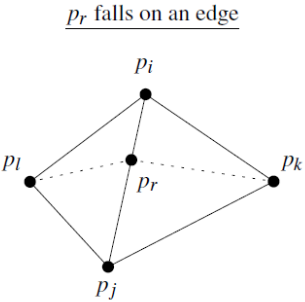
Randomized incremental approach





Pseudocode

```
13.         else (∗ p_r lies on an edge of p_i p_j p_k, say the edge p_i p_j ∗)
14.              Add edges from p_r to p_k and to the third vertex p_l of the other triangle that
                 is incident to p_i p_j, thereby splitting the two triangles incident to p_i p_j into
                 four triangles.
15.              LEGALIZEEDGE(p_r, p_i p_l, T)
16.              LEGALIZEEDGE(p_r, p_l p_j, T)
17.              LEGALIZEEDGE(p_r, p_j p_k, T)
18.              LEGALIZEEDGE(p_r, p_k p_i, T)
19. Discard p_−1 and p_−2 with all their incident edges from T.
20. return T
```

Randomized incremental approach

▸ Lemma 9.4

LEGALIZEEDGE($p_r, \overline{p_i p_j}, \mathcal{T}$)
1. (* The point being inserted is p_r, and $\overline{p_i p_j}$ is the edge of $\mathcal{T}$ that may need to be flipped. *)
2. **if** $\overline{p_i p_j}$ is illegal
3. **then** Let $p_i p_j p_k$ be the triangle adjacent to $p_r p_i p_j$ along $\overline{p_i p_j}$.
4. (* Flip $\overline{p_i p_j}$: *) Replace $\overline{p_i p_j}$ with $\overline{p_r p_k}$.
5. LEGALIZEEDGE($p_r, \overline{p_i p_k}, \mathcal{T}$)
6. LEGALIZEEDGE($p_r, \overline{p_k p_j}, \mathcal{T}$)

Randomized incremental approach

But

what about the correctness of algorithm?

Randomized incremental approach

Must show no illegal edge left behind!

- We see that every new edge added is incident to P_r .
- We will see that every new edge added is in fact legal.
- Together with the fact that an edge can only become illegal if one of its incident triangles changes,then our algorithm tests any edge that may become illegal.

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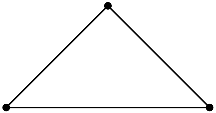
Randomized incremental approach

Lemma 9.10:

Every new edge created in 'DELAUNAYTRIANGULATION' or in 'LEGALIZEEDGE' during the insertion of P_r is an edge of the Delaunay graph of $\{p_{-1}, p_{-2}, p_0, ..., p_r\}$



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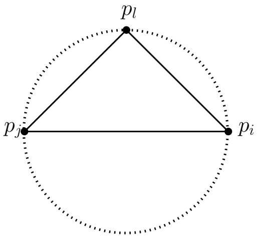

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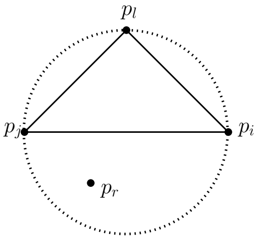

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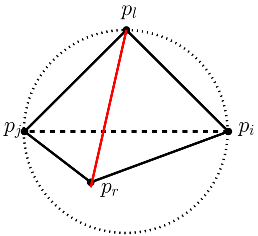

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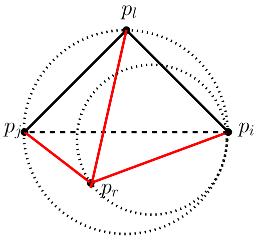

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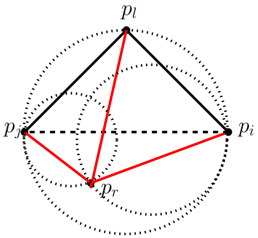

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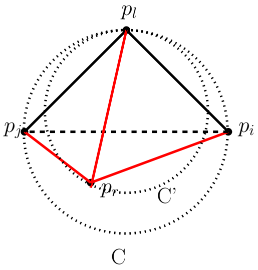

How to find the triangle containing the point p_r *?*

Go to algorithm

introduction DAG

✍ A point location structure D is a directed acyclic graph.

- The leaves of D correspond to the triangles of the current triangulation T
  ✓ exist cross-pointers between those leaves and the triangulation.
- The internal nodes of D correspond to triangles that have already been destroyed
  ✓ Any internal node gets at most three outgoing pointers.

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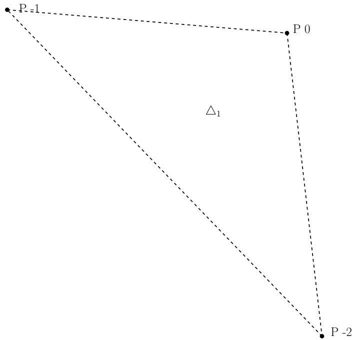
Constructing DAG



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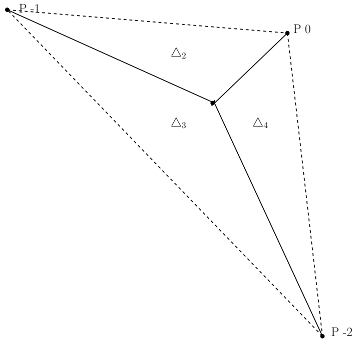

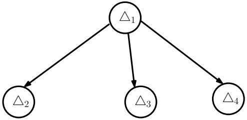

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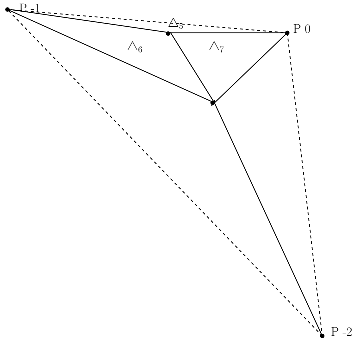

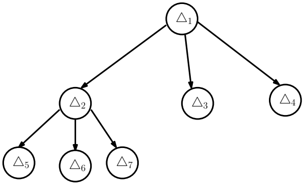

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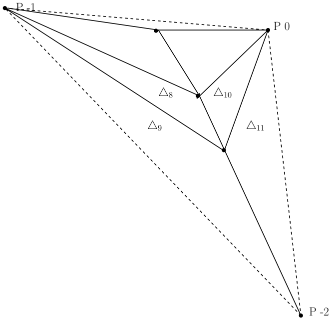

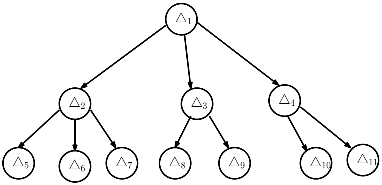

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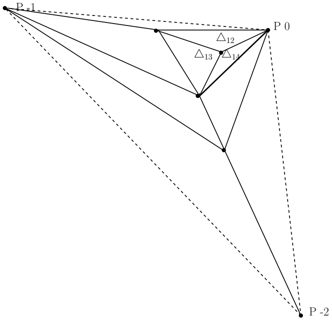

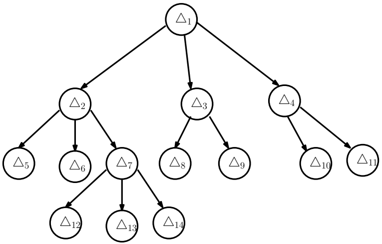

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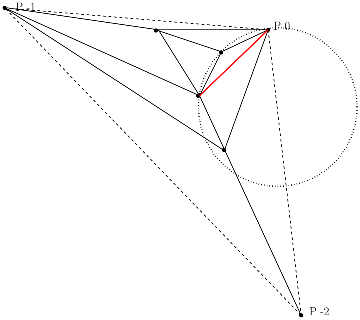

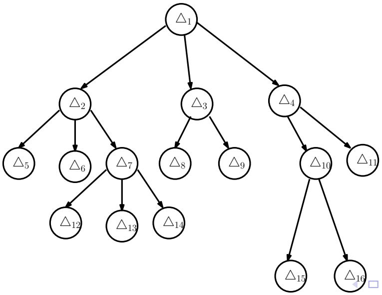

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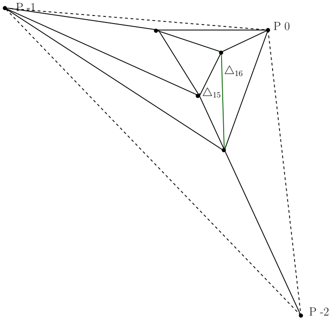
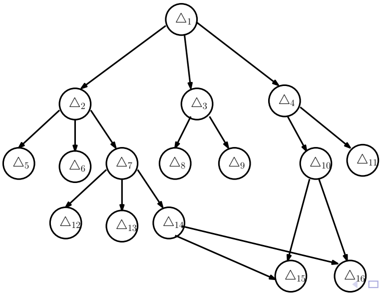




Point location

1. Start at the root of D,
2. Check the three children of the root and descend to the corresponding child,
3. check the children of this node,descend to a child whose triangle contains p_r,

   .

   .

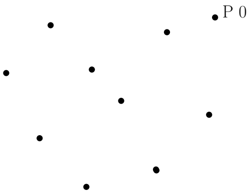
   .
4. until we reach a leaf of D,this leaf corresponding to a triangle in the current triangulation that contains p_r.

How to choose p_{-1} and p_{-2}?

and

How to implement the test of whether an edge is legal?

The first issue



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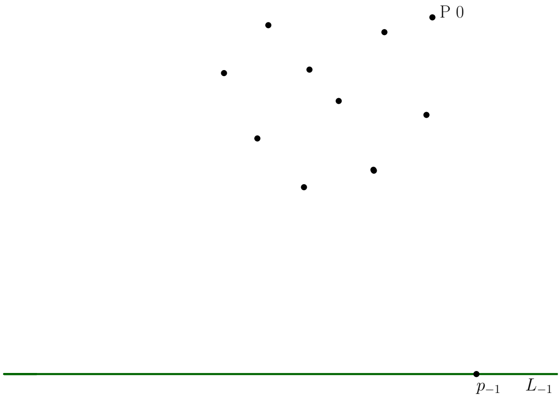

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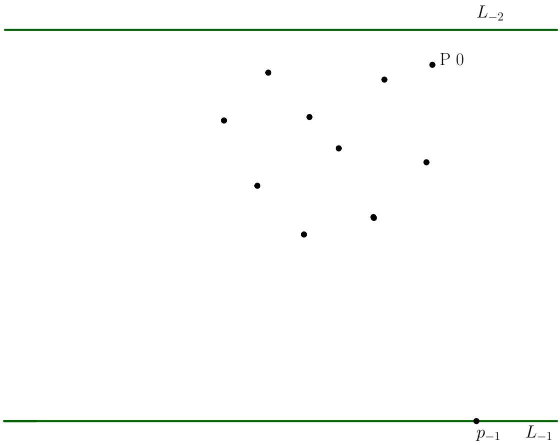

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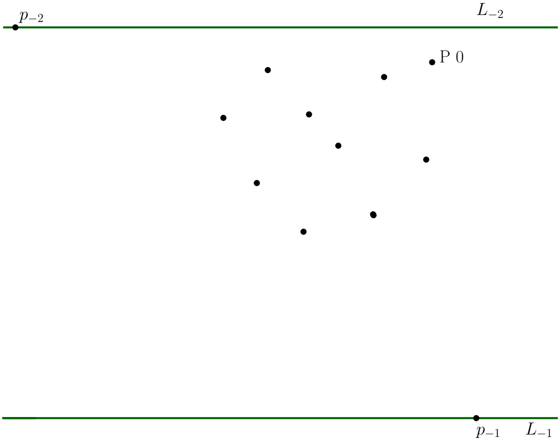

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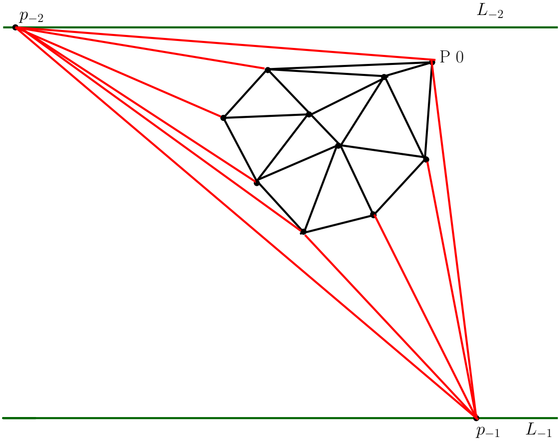

The first issue

Position of a point p_j with respect to the oriented line from p_i to p_k:

- p_j lies to the left of the line from p_i to p_{-1};
- p_j lies to the left of the line from p_{-2} to p_i;
- p_j is lexicographically larger than p_i.
  By our choice of p_{-1} and p_{-2}, the above conditions are equivalent.

The second issue

Let $\overline{p_i p_j}$ be the edge of to be tested,and let p_k and p_l be the other vertices of the triangles incident to $\overline{p_i p_j}$ (if they exist).

- $\overline{p_i p_j}$ is an edge of the triangle $p_0 p_{-1} p_{-2}$. These edges are **always** legal.
- The indices i,j,k,l are **all** non-negative.$\longleftarrow$ this case is normal
- All other cases $\overline{p_i p_j}$ is legal if and only if min(k,l) < min(i,j)

The Analysis

Lemma 9.11:

The expected number of triangles created by the algorithm is at most $9n + 1$.

Proof.

P_r := $\{p_1, p_2, ..., p_r\}$ $\qquad$ Dg_r := $Dg(\{p_{-2}, p_{-1}, p_0\} \cup P_r)$

- $\sharp$(new triangles in step r)$\leq$ $2k$ - 3 $\qquad$ $k{=}deg(p_r, Dg_r)$

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degree of p_r, over all possible permutations of the set P?

☞ *Backwards analysis:*

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- By Theorem 7.3: $\sharp$ Edges in $Dg_r \leq 3(r + 3) - 6$
- Total degree of the vertices in P_r < $2[3(r + 3) - 9] = 6r$
- The expected degree of a random point of $P_r \leq 6$
- we can bound the number of triangles created in step r:

$$E[\sharp\ (\triangle s \text{ in step } r)] \leq E\,[2deg(p_r, Dg_r) - 3]$$
$$= 2E[deg(p_r, Dg_r)] - 3$$
$$\leq 2 \times 6 - 3 = 9$$

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The Delaunay triangulation can be computed in $O(nlogn)$ expected time, using $O(n)$ expected storage.

Proof.

- Space follows from nodes in D representing triangles created, which by the previous lemma is $O(n)$.
- Not counting the time for point location,
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- Let $K(\triangle) \subset P$ be the points inside the circumcircle of a given triangle $\triangle$
- Therefore the total time for the point location steps is:

$$O(n + \sum_{\triangle} \text{card}(K(\triangle)))$$

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If P is a point set in general position,then

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Proof.

- P is in general position,then every subset P_r is in general position
- triangulation after insert p_r is the unique triangulation Dg_r
- T_r:= the set of $\triangle$s of Dg_r
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- Rewrite the sum:

$$\sum_{r=1}^{n}\Big(\sum_{\triangle\in T_r\setminus T_{r-1}} \text{card}(K(\triangle))\Big)$$

- Let $k(P_r, q)$ = $\sharp$ of triangles $\triangle \in T_r$; q $\in$ K($\triangle$)
- Let $k(P_r, q, p_r)$ = $\sharp$ of triangles $\triangle \in T_r$; q $\in$ K($\triangle$) , p_r is incident to $\triangle$
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- fix P_r, so k(P_r,q,p_r) depends only on p_r
- Probability that p_r is incident to a triangle is 3/r
- Thus:

$$E\,[\mathrm{k}(P_r,q,p_r)] \leq \frac{3k(P_r,q)}{r}$$

- Using:

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- Any of the remaining n-r points is equally likely to appear as p_{r+1}
- So:

$$E[\,k(P_r, p_{r+1})] = \frac{1}{n-r} \sum_{q \in P \setminus P_r} k(P_r, q)$$

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- $k(P_r, p_{r+1})$,
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- These are the triangles that will be destroyed when p_{r+1} is inserted. Theorem 9.6 (i)
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- By Theorem 9.1: T_m has 2(m + 3) - 2 - 3 = 2m + 1
- T_{m+1} has two triangles more than T_m
- Thus, card($T_r \setminus T_{r+1}$)
  $\leq$ card(triangles destroyed by p_{r+1})
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The Analysis

- Remember we fixed P_r earlier
- Consider all P_r by averaging over both sides of the inequality, but the inequality comes out identical.
- $E[\sharp$ of triangles created by $p_r]$
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- Therefore:

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- If we sum this over all r, we have shown that:

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polyhedral terrain made! ☺